**§9. Balance Equations across a Surface of Discontinuity**

In various continuum models, distributions of density, momentum, velocity, internal energy, stress, temperature, and other physical properties in gases, liquids, solids, and other media are represented by functions that are discontinuous across surfaces. The concepts of surface of discontinuity and piecewise continuous field prove to be very productive and helpful tools in modeling processes at phase boundaries as well as explosions and impacts.

In continuum mechanics, a distinction is made between strong and weak discontinuities.

***Definition.*** A surface across which all field quantities are continuous while some of their spatial or time derivatives have a jump is called a *weak discontinuity*.

***Definition.*** A surface across which some field quantities (density, velocity, stress, temperature, etc.) have a jump is called a *strong discontinuity*.

The geometry and kinematics of a surface of discontinuity are generally not known a priori and are to be determined by solving appropriate equations. Balance equations across a strong discontinuity are derived as follows.

Consider the neighborhood of an arbitrary point Р on a (generally moving) strong discontinuity surface  (Fig. 3.9.1). By the definition of strong discontinuity, some variables used in continuum mechanics have different values on opposite sides of . Denote the values of density, velocity, stress, internal energy, etc. on one side of  by superscript (1) (ρ(1), **v**(1), , *u*(1), ...) and those on the other side by superscript (2) (ρ(2), **v**(2), , *u*(2), ...). Then ρ(1) ≠ ρ(2), or **v**(1) ≠ **v**(2), or  ≠ , etc. at point Р. Accordingly, the regions on opposite sides of the surface are referred to as 1 and 2, respectively. Note, however, that all of the fields ρ(1), **v**(1), , *u*(1), ... , ρ(2), **v**(2), , *u*(2), ... vary continuously over the surface .

P

**n**(1)

1

**n**(2) = **n**



**n**(1)

Fig. 3.9.1. Schematic of a discontinuity surface.

At each point on  (e.g., at point Р in Fig. 3.9.1), define **n**(1) as the outward unit normal with respect to region 1 and **n**(2) as the outward unit normal with respect to region 2; i.e.,

**n**(1) = – **n**(2). (3.9.1)

Define oriented surfaces  and  as the surface  with outward unit normal vectors **n**(1) and **n**(2) assigned to each point, respectively.

Define the normal velocity *N*(1) of  as the rate of displacement of  along the normal **n**(1) and the normal velocity *N*(2) of  as the rate of displacement of  along the normal **n**(2) (see (3.1.3.) and (3.1.17)).

Hereinafter, each point on  is assigned a unique normal **n** defined as the normal vector pointing to region *1* (i.e., outwards with respect to region *2*) and a normal velocity *N* defined as the surface velocity component in the direction of **n**. Thus, it is assumed that  = , and

**х** ∈ : **n** = **n**(2) = – **n**(1), *N* = *N*(2) = − *N*(1), *N*(1) **n**(1) = *N*(2)**n**(2). (3.9.2)

The normal **n** = **n**(2) = – **n**(1) is also used to define the normal components of velocities **v**(1) and **v**(2), of tractions  and , of heat flux vectors **q**(1) and **q**(2), and of generalized surface forces and fluxes on opposite sides of the discontinuity, denoted by subscript *n*:

*=**nk*, = *nk*, *=**nk*. *=**nk* (*i =* 1, 2). (3.9.3)

**Eulerian control volume around a discontinuity**. Consider an arbitrary Eulerian volume *V*E intersected by a discontinuity (see Fig. 3.9.2). The surface  divides *V*E into two volumes, *V*(1) and *V*(2), which vary with time as  moves. The intersection of  with the boundary surface *S*Eof *V*E is a closed curve *L* dividing *S*E into two parts, *S*(1) and *S*(2), which vary with time while *S*E does not*.* In summary, we have

*V*E = *V*(1)(*t*) ∪ *V*(2)(*t*), *S*E = *S* (1)(*t*) ∪ *S* (2)(*t*). (3.9.4)

Fig. 3.9.2. Schematic of an Eulerian control volume *V*E around discontinuity surface .



**n(1)**

**n(2)** =**n**

*V*2

*V*1

*S*2

The volumes *V*(1) and *V*(2) are bounded by *S*(1)∪ and *S*(2)∪ , respectively.

**Additivity, classical discontinuities, surface phase.** In most cases, it can be assumed that the mass, momentum, and internal energy in the volume *V*E = *V*(1)(*t*)∪*V*(2)(*t*) with boundary *S*E = *S*(1)(*t*)∪*S*(2)(*t*) intersected by a surface of discontinuity  are additive, and so are the effects produced by electromagnetic fields and forces acting within the volume and on its boundary:

**d*V* = ** d*V* + ** d*V*, ****v** d*V =***v** d*V +***v** d*V*,

 =  + , (3.9.5)

**F** d*V = ***F** d*V +***F** d*V*, d*V = *d*V +*d*V*,

*u* d*V* = *u* d*V* + *u* d*V*,

 =  + ,

 =  +.

This assumption implies that  does not contribute to the mass, momentum, and energy contained in *V*E or transferred across *S*E. In other words, there is no mass, momentum, energy, or force localized on the singular surface  or on its intersection *L* with *S*E.

***Definition.*** A surface of discontinuity is called a *classical discontinuity* if it does not contribute to the mass, momentum, and energy of the medium and neither mass, nor momentum, nor energy are transferred across its boundary curve.

Discontinuities of more general type are discussed in §12 below.

**Integral balance across a classical discontinuity.** Consider an arbitrary Eulerian control volume *V*E intersected by a classical discontinuity . According to conservation laws, there is no internal production of mass, linear momentum, angular momentum, or total energy within the volume ( = 0). For a volume *V*E divided into *V*(1)(*t*) and *V*(2)(*t*), with boundary *S*E divided into *S*(1)(*t*) and *S*(2)(*t*), balance equation (3.8.18) with  = 0 is written as

*f* d*V* + (3.9.6)

=ρ *vk f* + ψ*k*) *nk*d*S* + ρ *vk f* + ψ*k*) *nk* d*S +*d*V* +**d*V*.

Since all functions here are continuously differentiable within *V*(1) and *V*(2), the time derivatives of integrals over moving volumes *V*(*i*)(*t*) on the left-hand side of this equation can be expressed by using theorem (3.1.5) and noting that the respective boundaries of these volumes are the surfaces *S*(*i*)∪(*i* = 1, 2):

*f* d*V* = (ρ *f*)d*V* +*f N* d*S* +*f N* d*S*,

*f* d*V* = (ρ *f*)d*V* +*f N* d*S* +*f N* d*S*. (3.9.7)

The normal velocity *N* of a stationary surface *S*E is zero everywhere, including its parts *S*(1)(*t*) and *S*(2)(*t*); i.e.,

*х* ∈ *S*(1), *х* ∈ *S*(2): *N* = 0, (3.9.8)

Therefore, the second integrals on the right-hand sides in (3.9.7) vanish. According to (3.9.2), the normal velocities of and  are *N*(1) and *N*(2), respectively:

*х* ∈ : *N* = *N* (1), *х* ∈ : *N* = *N*(2). (3.9.9)

Expressions for the volume integrals of ∂(ρ*f*)*/*∂*t* are found by using the generalized balance equation in (3.8.20), which holds everywhere within *V*(1) and *V*(2) except on the surfaces  and :

(ρ *f* ) = ∇*k*(– ρ *f vk +* ψ*k*) + ρ . (3.9.10)

The integral of this equation over *V*(1) or *V*(2) is

(ρ *f* )d*V* = (– ρ *f vk +* ψ*k*)d*V* + d*V* (*i =* 1, 2). (3.9.11)

Since the integrand in the first term on the right-hand side is a divergence, Gauss–Ostrogradsky theorem (3.1.25) can be used to transform the volume integral into a surface integral over the boundary surface *S*(*i*)∪ of *V*(*i*) (*i* = 1, 2):

(ρ *f* )d*V* = ρ *f vk* + ψ*k*) *nk* d*S* +  d*V* (*i =* 1, 2). (3.9.12)

Writing out the resulting expressions for *i =* 1 and *i =* 2,

(ρ *f* )d*V* = ρ *f vk* + ψ*k*) *nk* d*S* +ρ *f vk* + ψ*k*) *nk* d*S* + d*V*,

(3.9.13)

(ρ *f* )d*V* =  ρ *f vk* + ψ*k*) *nk* d*S* +ρ *f vk* + ψ*k*) *nk* d*S* + d*V*,

we represent Eqs. (3.9.7) as

*f* d*V* = ρ *f vk* + ψ*k*) *nk* d*S* (3.9.14)

+ ρ *f* (*vk* –  + ψ*k*) ** d*S* + d*V*,

*f* d*V* =ρ *f vk* + ψ*k*) *nk* d*S*

+ ρ *f* (*vk* – ) + ψ*k*) d*S* +d*V*.

Subtracting these equations from (3.9.6), we obtain an integral equation over the surface of discontinuity  across which interaction occurs between the media in *V*(1) and *V*(2) as part of internal processes in *V*:

 +  = 0.

(3.9.15)

**Discontinuity-fixed coordinates.** In addition to the observer's coordinate system *x*1*x*2*x*3*t* (denoted by K), where the velocity field is **v**(**x**, t) and the local normal velocity of the discontinuity is *N***n**, a discontinuity-fixed system  (denoted by K′) is defined at each point on the surface of discontinuity. During the infinitesimal time interval from *t* до *t* + d*t*, the discontinuity-fixed reference frame moves with the velocity *N***n**, which implies the following transformation rules for coordinates *xk*,  and velocities **v**, **v**′:

 = *xk*  − *Nnk* (*t* – ),  = *vk* − *Nnk*, *N*′ = 0,

(*N* = *N*(2) = – *N*(1), *Nnk* = = ). (3.9.16)

Then subscripts (1) and (2) referring to field quantities on opposite sides of  are used to rewrite Eq. (3.9.15) as

d*S* = 0. (3.9.17)

Since the integral vanishes over any portion of  and the functions in the integrand vary continuously over , a zero integrand theorem analogous to (3.1.27) applies to the surface integral; i.e., the integrand in (3.9.15) is zero at each point on :

(− ρ(2) *f*(2)  + ) *nk* – (– ρ(1) *f*(1)  + )*nk* = 0

or, equivalently,

(– ρ(1) *f*(1)  + )  + (– ρ(2) *f*(2)  + )  = 0. (3.9.18)

The same equation rewritten in terms of mass fluxes ξ(1) and ξ(2)  is

ξ(1) *f*(1) +  + ξ(2) *f*(2) + = 0 (3.9.18a)

(**v**′(1) = **v**(1) – *N*(1)**n**(1) = **v**(1) – *N***n**, **v**′(2) = **v**(2) – *N*(2)**n**(2) = **v**(2) – *N***n**

ξ(α) = – ρ(α) (α = 1, 2)).

For *f* = 1 and ψ*k* = 0, this yields the mass conservation equation

ξ(1)+ ξ(2) = 0, (3.9.18b)

which implies that the mass flux ξ along the normal vector defined as **n** = **n**(2) = – **n**(1) (see (3.9.2) and (3.9.3)) is continuous across the surface of discontinuity:

ξ = ξ(2) = − ξ(1), (3.9.18c)

ξ(2) = – ρ(2)*nk* = – ρ(2), ξ(1) = – ρ(2) = ρ(1)*nk*

(*nk* =  = – ).

Instead of (3.9.15), the starting point for deriving an entropy balance equation (with *f* = *s* and ψ*k* = – *qk/T*) should be an equation allowing for nonnegative entropy production not only in the bulk, with source strength Φ (see (3.7.4), (3.7.5)), but also on the discontinuity surface , with source strength ΨS:

d*V* + (3.9.19)

=ρ *vks* − )d*S* + ρ *vk s* − )d*S*

*+*d*V* +**d*V*

+ , ΨS ≥ 0.

Then, with (3.9.10) replaced by local balance (3.8.15), the above derivation is repeated to obtain a balance equation allowing for surface entropy production:

(− ρ(2)*s*(2) − /*T*(2))  + (− ρ(1)*s*(1) − /*T*(1))  + ΨS = 0. (3.9.20)

In summary, balance equations (3.9.18)–(3.9.20) lead to the following theorem.

**Theorem.** In discontinuity-fixed coordinates K′ (*N*′ = 0, see (3.9.16)), local mass, linear momentum, angular momentum, and total energy conservation, as well as local entropy balance, can be represented in terms of the generalized quantities defined in (3.8.17) at each point on a classical discontinuity surface:

ξ ( *f*(2) – *f*(1)) = − ( – ) + , (3.9.21)

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| *F* | **v**′ | (**x** × **v**′) + **m** | *u* + l/2(*v*′)2 | *S* |
| ψ*n* | **σ** *n* | (**x** × **σ***n*) + **θ***n* | (**σ***n ·* **v**′) – *qn* | – */T* |
|  | 0 | 0 | 0 | ΨS |

where **v**′ = **v** – *N***n** = **v** – *N*(1)**n**(1) = **v** – *N*(2)**n**(2) is the velocity of the medium in the discontinuity-fixed system K′ at that point[[1]](#footnote-1).

This equation imposes constraints on the values of ρ, **v**′, **m**, σ*kl*, **q**, and **θ***k* at each point on opposite sides of a discontinuity.

Note that the bulk external sources represented by Ω(ext) in (3.8.17), including external forces (**F**), the corresponding moment combined with body couples ([**x** × **F**] + **M**), and the rate of change in energy due to external work and energy conversion between electromagnetic and internal energy (**F ⋅ v** + ), do not contribute to the balance equations across a discontinuity.

Substituting expressions from the table into the equation in (3.9.21), we obtain mass, linear momentum, angular momentum, total energy, and entropy balance equations across a discontinuity:

− ρ(2) = − ρ(1) ≡ ξ, (3.9.22)

ξ(**v**′(2) – **v**′(1)) = − ( – ), (3.9.23)

ξ

= **−  + **, (3.9.24)

ξ

= − , (3.9.25)

ξ(*s*(2) – *s*(1)) =  –  + ΨS. (3.9.26)

Here, **x**(2) = **x**(1) = **x**, and all quantities are assigned their values at the same point on either side of the surface.

Note that assuming **n** ≡ **n**(2) ≡ –**n**(1) implies that a positive mass flux across the discontinuity,

ξ = – ρ(1)(**v**(1) – *N***n**) **⋅ n** = – ρ(2)(**v**(2) – *N***n**) **⋅ n**, (3.9.27)

is from region *1* to region *2*. Vice versa, a negative value of the flux corresponds to mass flow from (2) to (1). Since ΨS > 0, it follows from Eq. (3.9.26) that zero heat flux ( =  = 0) implies an increase in entropy if the discontinuity is crossed from region *1* into region *2* (*s*(2) > *s*(1) if ξ > 0). Similarly, entropy increases across the surface if mass flows from *2* to *1* (*s*(1) > *s*(2) if ξ < 0).

The change in a quantity across a discontinuity is represented by the jump operator

[ϕ] ≡ ϕ(2) – ϕ(1). (3.9.28)

In particular,

[**v**′] ≡ **v**′(2) – **v**′(1) = **v**(2) – **v**(1) ≡ [**v**],  ≡  – ,

 ≡ ,

 ≡ ,

[**m**] ≡ **m**(2) – **m**(1), [] ≡  – ,

where brackets should not be confused with those denoting the cross product [**a**× **b**] ≡ (**a**× **b**) of two vectors. Then Eqs. (3.9.22)–(3.9.26) can be rewritten in compact notation as

−  ≡ [ξ] = 0 (ξ = − ρ(1) = − ρ(2)), (3.9.29)

ξ [**v**] = −  ( [**v**] = ), (3.9.30)

ξ = − , (3.9.31)

ξ [**m**] = − [], (3.9.32)

ξ [*s*] =  + ΨS, ΨS ≥ 0. (3.9.33)

Angular momentum equation (3.9.32) is derived from (3.9.25) by subtracting the equation

ξ = – ,

which is the cross product between position vector **x** and momentum equation (3.9.23).

Balance of intrinsic angular momentum is not considered here, because intrinsic spin is irrelevant to most problems of interest, and it can therefore be assumed that **m** = 0 and  = 0.

Equations (3.9.29)–(3.9.33) can be written in terms of velocities in the observer's frame. According to (3.9.16), it holds that

**v**′(α) = **v**(α) – *N***n**,  =  – *N* (α =1, 2).

Therefore, Eqs. (3.9.29)–(3.9.33) become

− ρ(1) ( – *N*) = − ρ(2) ( – *N*) ≡ ξ,

ξ [**v**] = − , (3.9.34)

ξ  = − ,

ξ [*s*] =  + ΨS , ΨS ≥ 0.

To derive the third (energy) equation here, rewrite (3.9.31) as

ξ  = ,

and use the definition of jump operator [ ] in (3.9.28):

ξ

=  –  − .

After simple rearrangements, this equation becomes

ξ

= .

Rewriting the second (linear momentum) equation in (3.9.34) as

ξ (**v**(2) – **v**(1)) = − ( – ), (3.9.35)

and using definition (3.9.28) for jump operator [ ] yields the third (energy) equation in (3.9.34).

**Laboratory frame coordinates.** In addition to arbitrary (observer's) and discontinuity-fixed coordinate systems K and K′, a laboratory coordinate system *K*\* can be employed, where the medium ahead of the moving surface of discontinuity is at rest. In this frame of reference, the normal **n** is directed into the region at rest, referred to as region *1*, and velocities are labeled with an asterisk \*. The normal velocity of the discontinuity in *K*\*, sometimes called *propagation speed*, is denoted by *D*:

*v*\*(1) = 0,  = 0, *N*\* ≡ *D* > 0 ( =  – *D* < 0, α = 1, 2). (3.9.36)

In the laboratory system,

ξ ≡ − ρ(1) ( – *N*) = − ρ(1) ( – *N*\*) = ρ(1)*D*,

and the balance equations in (3.9.34) read

ξ = ρ(1)*D* = ρ(2) (*D* – ) (**v**\* ≡ **v**\*(2) ≡ [**v**]),

ρ(1)*D* **v**\* = − ,

ρ(1) *D*(*u*(2) – *u*(1) + 1/2 (*v*\*)2) = – **⋅ v**\* + , (3.9.37)

ρ(1)*D*[*s*] =  + ΨS, ΨS ≥ 0.

**Contact discontinuity.** The set of all possible surfaces of discontinuity includes *contact discontinuities*, which are defined as those with zero mass flux across the surface (ξ = 0). Under this condition, the following relations between velocities on opposite sides hold in coordinate systems *K*′, *K*, and *K*\*, respectively:

 =  = 0,  =  = *vn = N*,  =  =  *= D* = 0. (3.9.38)

Contact discontinuities separate distinct phases. Examples include liquid–gas, liquid–liquid, and solid–solid interfaces, as well as the boundary between a solid body and the fluid flowing past it.

As applied to contact discontinuities, Eqs. (3.9.34) have the form

 = 0 ( = ),

 = 0 (**⋅** **v**(2) –  = **⋅** **v**(1) – ), (3.9.39)

[] = 0,

 + ΨS = 0, ΨS ≥ 0.

Thus, normal component *vn*, normal traction **σ***n*, and surface couple vector **θ***n* are continuous across contact discontinuities,

 =  = *vn* = *N*,  =  = , (3.9.40)

 =  = ,

whereas density ρ, tangential velocity **v**τ (**v**(α) = **n** + , α = 1, 2), and heat flux *qn* may undergo a jump. Note that the jumps [*qn*] and [**v**τ] are related by the second (energy) equation in (3.9.39):

 =  ⋅ ,

 = **⋅** ( – ) = **⋅**( – ) (3.9.41)

( = **v**(α) – *nk***n**,  =  – σ*nn* **n**, α = 1, 2).

The change in energy due to the work done by the shear force  on a slip surface always goes to heat (**⋅** ( – ) > 0). Thus, a slip discontinuity acts as a surface heat source, and entropy increases across the discontinuity.

1. Cross product is written in parentheses (i.e., (**а×b**) ≡ [**а×b**]) since brackets are used to denote a different operator in this chapter. [↑](#footnote-ref-1)